QUADRATIC EQUATIONS

Solving quadratic equations can be difficult, but luckily there are several different methods that we can use depending on what type of quadratic that we are trying to solve. The four methods of solving a quadratic equation are factoring, using the square roots, completing the square and the quadratic formula.

Solving Equations

The standard form of a Quadratic equation is $\mathbf{ax^2 + bx + c = 0}$ whereby a, b, c are known values and 'a' can't be 0. 'x' is a variable (we don't know it yet). a is the coefficient of x^2 , b is the coefficient of x and c is a constant term. Quadratic equation is also called an equation of degree 2 (because of the 2 on x). There are several methods which are used to find the value of x. These methods are:

1. by Factorization
2. by completing the square
3. by using quadratic formula

The Solution of a Quadratic Equation by Factorization

Determine the solution of a quadratic equation by factorization

We can use any of the methods of factorization we learnt in previous chapter. But for simplest we will factorize by splitting the middle term. For Example: solve for x, $x^2 + 4x = 0$

solution

Since the constant term is 0 we can take out x as a common factor.

So, $x^2 + 4x = x(x + 4) = 0$. This means the product of x and $(x + 4)$ is 0. Then, either $x = 0$ or $x + 4 = 0$. If $x + 4 = 0$ that is $x = -4$. Therefore the solution is $x = 0$ or $x = -4$.

Example 1

Solve the equation: $3x^2 =- 6x – 3$.

first rearrange the equation in its usual form.

that is:

$3x^2 = - 6x – 3$

$3x^2 + 6x + 3 = 0$

now, factorize the equation by splitting the middle term. Let us find two numbers whose product is 9

and their sum is 6. The numbers are 3 and 3. Hence the equation $3x^2 + 6x + 3 = 0$ can be written as:

$3x^2 + 3x + 3x + 3 = 0$

3x(x + 1) + 3(x + 1) = 0

(3x + 3)(x + 1) (take out common factor which is (x + 1))

either (3x + 3) = 0 or (x + 1) = 0

therefore 3x = -3 or x = -1

x = -1 (divide by 3 both sides) or x = -1

Therefore, since the values of x are identical then x = -1.

Example 2

solve the equation 10 – 3*y* - 1 = 0 by factorization.
Solution

Two numbers whose product is -10 and their sum is -3 are 2 and -5.

Then, we can write the equation $10y^2 – 3y - 1 = 0$ as:

2*y*(5*y* + 1) – 1(5*y* + 1) = 0

(2*y* – 1)(5*y* + 1) = 0

Therefore, either $2y - 1 = 0$ or $5y + 1 = 0$

$y = \frac{1}{2}$ (divide by 2 both sides) or $y = -\frac{1}{5}$ (divide by 5 both sides).

$y = \frac{1}{2}$ or $y = -\frac{1}{5}$

Example 3

solve the following quadratic equation by factorization: $4x^2 - 20x + 25 = 0$.

Solution

We need to split the middle term by the two numbers whose product is 100 and their sum is -20. The numbers are -10 and -10.

The equation can be written as:

$4x^2 - 10x - 10x + 25 = 0$

$2x(2x - 5) - 5(2x - 5) = 0$

$(2x - 5)(2x - 5)$ (take out common factor. The resulting factors are identical. This is a perfect square)

since it is a perfect square, then we take one factor and equate it to 0. That is:

$2x - 5 = 0$

$2x = 5$ then, divide by 2 both sides.

Therefore

$x = \frac{5}{2}$.

Example 4

solve the equation $x^2 - 16 = 0$.

Solution

We can write the equation as x^2 - 4^2 = 0. This is a difference of two squares. The difference of two squares is an identity of the form:

$a^2 - b^2 = (a - b)(a + b)$.

So, $x^2 - 4^2 = (x - 4)(x + 4) = 0$

Now, either $x- 4 = 0$ or $x + 4 = 0$

Therefore $x = 4$ or $x = -4$

The Solution of a Quadratic Equation by Completing the Square

Find the solution of a quadratic equation by completing the square

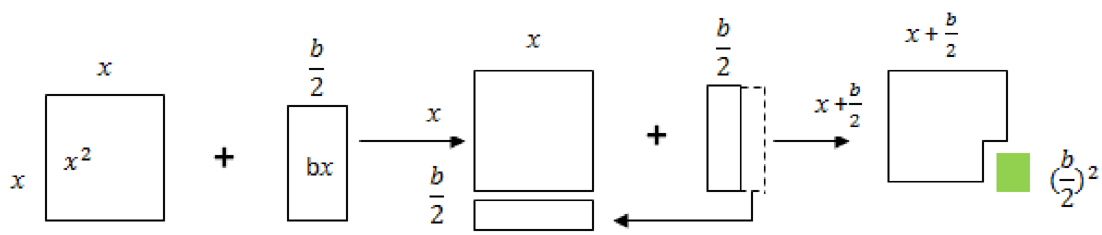
Completing the square.

Completing the Square is where you take a Quadratic Equation like: $ax^2 + bx + c = 0$ and turn it into: $a(x + d)^2 + e = 0$ whereby $d = \frac{b}{2a}$ and $e = c - \frac{b^2}{4a}$

How to complete the square

If we have a simple expression like $x^2 + bx$ having x twice in the same expression can make life hard. What can we do?

See an illustration below:



As you can see $x^2 + bx$ can be rearranged nearly into a square and we can complete the square with $(\frac{b}{2})^2$

In Algebra it looks like this:

$$x^2 + bx \quad + (\frac{b}{2})^2 = (x + \frac{b}{2})^2$$

So, by adding $(\frac{b}{2})^2$ we can complete the square

And $(x + \frac{b}{2})^2$ has x only once, which is easier to solve.

A general Quadratic Equation can have a coefficient of 'a' in front of x^2. i.e. $ax^2 + bx + c = 0$. How to complete the square?

Step 1: divide all term by a (coefficient of x^2)

i.e. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step 2: move the number term $(\frac{c}{a})$ to the right side of the equation.

i.e. $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3: complete the square on the left side of the equation and balance this by adding the same value to the right side of the equation.

i.e. $x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$

which is the same as:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Step 4: Take the square root on both sides of the equation

i.e. $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

Step 5: Subtract the number that remains on the left side of the equation to find x

i.e. $x = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} - \frac{b}{2a}$

$$x = \pm\frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5

Add a term that will make the following expression a perfect square: $x^2 - 8x$

Since the coefficient of x^2 is 1, then will add a constant term $(\frac{b}{2})^2$,

where by b = -8

Therefore the term to be added is: $(\frac{b}{2})^2 = (\frac{8}{2})^2 = 4^2 = 16$

Our expression will be $x^2 - 8x + 16$

find a term that must be added to make the following expression a perfect square: $x^2 + 10x$

Solution

The coefficient of x^2 is1, so we will add the term $(\frac{b}{2})^2$, whereby b = 10

Therefore, the term to be added is: $(\frac{b}{2})^2 = (\frac{10}{2})^2 = 25$.

Our expression will be $x^2 + 10x + 25$.

Example 6

solve the following quadratic equation by completing the square: $x^2 + 4x + 1 = 0$

Solution

Step 1: the step can be skipped since the coefficient of x^2 is 1

Step 2: move the number term to the right

$x^2 + 4x = -1$

Step 3: complete the square on the left side of the equation and balance this by adding the same number on the right of the equation.

$(\frac{b}{2})^2 = (\frac{4}{2})^2 = 4$

$x^2 + 4x + 4 = -1 + 4$

$(x + 2)^2 = 3$

Step 4: take the square root on both sides of the equation

$x + 2 = \pm\sqrt{3}$

Step 5: subtract 2 both sides

$x = \pm\sqrt{3} - 2$

$x = -0.268$ or $x = -3.732$

Example 7

solve by completing the square: $3x^2 + 7x – 6 = 0$

Solution

Step 1: divide each term by 3

$x^2 + \frac{7}{3}x - 2 = 0$

Step 2: move the number term to the right

$x^2 + \frac{7}{3}x = 2$

Step 3: complete the square on the left side of the equation and add the same number to the right of the equation to balance this.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{7}{2 \times 3}\right)^2 = \frac{49}{36}$$

$$x^2 + \frac{7}{3}x + \frac{49}{36} = 2 + \frac{49}{36}$$

$$\left(x + \frac{7}{2}\right)^2 = \frac{121}{36}$$

Step 4: take square root both sides

$$x + \frac{7}{2} = \pm\sqrt{\frac{121}{36}}$$

$$x + \frac{7}{2} = \pm\frac{11}{6}$$

Step 5: subtract $\frac{7}{2}$ from both sides

$$x = \pm\frac{11}{6} - \frac{7}{2}$$

$$x = -\frac{5}{3} \text{ or } x = -\frac{16}{3}$$

General Solution of Quadratic Equations

The Quadratic Formula

Derive the quadratic formula

The special quadratic formula used for solving quadratic equation is:

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}.$$

Just plug in values of a, b, and c and then do calculations.

The symbol $\pm$ means there are two answers, which are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and}$$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

But sometimes we don't get two answers, and the discriminate tells why? The term $b^2 - 4ac$ is called Discriminant because It can discriminate between the possible types of answers.

- When $b^2 - 4ac$ is positive we get two real solutions.
- When $b^2 - 4ac$ is zero we One real solution (both answers are the same)
- When $b^2 - 4ac$ is negative we get two complex solutions (not real solutions). We are not going to learn about this, is not in our level.

Quadratic Equations using Quadratic Formula

Solve quadratic equations using quadratic formula

Example 8

solve $5x^2 – 8x + 3 = 0$ by using quadratic formula.

Solution

Recall quadratic formula: $x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$

from the question; a = 5, b = -3 and c = 3

then $x = \frac{-(-8) \pm\sqrt{(-8)^2 - 4 \times 5 \times 3}}{2 \times 5}$

$$x = \frac{3 \pm\sqrt{64-60}}{10}$$

$$x = \frac{3 \pm\sqrt{4}}{10}$$

$$x = \frac{3 \pm 2}{10}$$

either $x = \frac{3+2}{10}$ or $x = \frac{3-2}{10}$

Therefore $x = \frac{1}{2}$ or $x = \frac{1}{10}$

Example 9

solve this quadratic equation by using quadratic formula: $3x^2 = - 7x - 4$

Solution

First, rearrange the equation, move all the terms on the right side to the left side of the equatior

$3x^2 + 7x + 4 = 0$

now, recall the quadratic formula: $x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$

from our equation; a = 3, b = 7, and c = 4

then plug in the values of a, b, and c and do calculations

$$x = \frac{-7 \pm\sqrt{7^2 - 4 \times 3 \times 4}}{2 \times 3}$$

$$x = \frac{-7 \pm\sqrt{49 - 48}}{6}$$

$$x = \frac{-7 \pm\sqrt{1}}{6}$$

$$x = \frac{-7 \pm 1}{6}$$

either $x = \frac{-7+1}{6}$ or $x = \frac{-7-1}{6}$

Therefore, x = -1 or $x = -\frac{4}{3}$

Word problems leading to quadratic equations

Given a word problem; the following steps are to be used to recognize the type of equation.

Step1: choose the variables to represent the information

Step 2: formulate the equation according to the information given

Step 3: solve the equation by using any of the method you know

In order to be sure with your answers, check if the solution you obtained is correct.

Example 10

the length of a rectangular plot is 8 centimeters more than the width. If the area of a plot is $240cm^2$, find the dimensions of length and width.

Solution

Let the width be x

The length of a plot is 8 more than the width, so the length of a plot be x + 8

We are given the area of a plot = $240cm^2$ and the area of a rectangle is given by length ×width

then $(x + 8) \times x = 240$

$x^2 + 8x = 240$

rearrange the equation

$x^2 + 8x – 240 = 0$

then solve the equation to find the value of x

Solving by splitting the middle term, two numbers whose product is -240 and their sum is 8, the number

are -12 and 20

our equation becomes; $x^2 + 20x – 12x – 240 = 0$

$x(x + 20) – 12(x + 20) = 0$

either $(x - 12) = 0$ or $(x + 20) = 0$

$x = 12$ or $x = -20$

since we don’t have negative dimensions, then the width is 12cm and the length is 12 + 8 = 20cm

Therefore the rectangular plot has the length of 20cm and the width of 12cm.

Example 11

A piece of wire 40cm long is cut into two parts and each part is then bent into a square. If the sum of the areas of these squares is 68 square centimeters, find the lengths of the two pieces of wire.

Solution

We don't know the lengths of the two pieces, so let one of the length be l.

then the length of the other piece of wire will be $40 - l$

each piece of wire formed a square. A square has four equal lengths.

A piece of wire with a length of $l$cm will form a square having length of $\frac{l}{4}$ cm each side and its area will be length $\times$ length = $(\frac{l}{4})^2$

A piece of wire with a length of $(40 - l)$cm will form a square having length of $(\frac{40-l}{4})$cm each side and its area will be length $\times$ length = $(\frac{40-l}{4})^2$

the sum of the areas = $(\frac{l}{4})^2 + (\frac{40-l}{4})^2$

$$68 = (\frac{l}{4})^2 + (\frac{40-l}{4})^2$$

$$68 = \frac{l^2}{16} + \frac{1600-80l+l^2}{16}$$

$$68 = \frac{l^2+1600-80l+l^2}{16}$$

$68 \times 16 = 2l^2 - 80l + 1600$ (multiply by 16 both sides)

$2l^2 - 80l + 1600 = 1088$

$2l^2 - 80l + 512 = 0$ (subtract 1088 both sides)

$l^2 - 40l + 256 = 0$ (divide by 2 throughout)

solve the equation to find values of l

by splitting the middle term

$l^2 - 8l - 32l + 256 = 0$

$l(l - 8) - 32(l - 8) = 0$

$(l - 32)(l - 8) = 0$

either $(l - 32) = 0$ or $(l - 8) = 0$

$l = 32$ or $l = 8$

Therefore the two pieces have the length of 8cm and 32cm

Exercise 1

1. Solve each of the following quadratic equations by using factorization method:

1. $-6x^2 + 23x - 20 = 0$
2. $X^2 - x - 12 = 0$

2. Solve these equations by completing the square:

1. $X^2 - 11 - 3 = 0$
2. $X^2 - \frac{10}{9}x - \frac{2}{9} = 0$

3. What must be added to the following expression to make them complete squares?

1. $X^2 + 10x$
2. $X^2 - \frac{4}{5}x$

4. Use general formula for quadratic equations to solve the following quadratic equations:

1. $2x^2 - 7x + 3 = 0$
2. $4x^2 + 8x + 4 = 0$

5. Find two consecutive odd numbers whose product is 255.